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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE LONGITUDINAL VIBRATIONS OF A RUBBED STRING.

By Harvey N. Davis.

With Three Plates.

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§ 1. Introductory.

This work was originally undertaken as the first step in an investigation of the effect of vibration on magnetic quality. In the previous study of this subject, although many exact measurements of the magnetic phenomena had been made by a number of writers, in no case had the conditions been such as to admit of an equally exact description of the mechanical phenomena involved. It seemed desirable, therefore, to renew the attack upon the subject, using as the mechanical disturbance a steady state of vibration whose character was known, and whose intensity could be accurately measured and easily controlled. It was found that longitudinal vibrations suitable for this purpose could be maintained in a steel wire by means of a mechanically operated rubbing wheel, and a study of their nature has shown them to be identical in all respects with the familiar transverse vibrations of a violin string, so that the parallelism between longitudinal and transverse vibrations which the mathematics of the subject predicts, finds here a unique and exceedingly detailed experimental verification. This purely mechanical discussion has thus acquired a considerable interest in itself, and has, therefore, been carried far beyond the requirements of the original problem. It is now presented as a separate paper; the magnetic problem will be considered at another time.

In two respects the present paper goes beyond the published work on the violin string. In the first place, the fact that a given state of vibration can be maintained indefinitely and can be reproduced at will has made it possible, for the first time, to measure amplitudes under conditions which are accurately known. This has resulted in a good quantitative verification of the established theory, and especially of the law connecting amplitude with bowing speed. And in the second place, it is hoped that the graphical method which is here presented for solving the differential equation of vibrating strings, subject to the most general boundary conditions which the problem of the rubbed string requires, will prove to be a valuable addition to the theoretical side of the subject. It is possible that this part of the paper will have some interest for the pure mathematician, for it is rather unusual that graphics should be able to give, not simply an approximate, but an exact and rigorous solution of a problem of such complexity. A striking instance of this kind is, of course, Euler's solution of the problem of the plucked string.

As has been indicated, all previous work on rubbed strings has been concerned with transverse vibrations. In this form the problem is fully as old as that of the plucked or of the struck string, but it differs essentially from them in that it is not easy to set up, *à priori*, a satisfactory analytical statement of the action of a bow, as was attempted by the earlier mathematicians.[1] The first results of any importance are due to Thomas Young,[2] who formulated for plucked strings, and less definitely for bowed strings,

YOUNG'S LAW: *No overtone is present which would have a node at the point of excitation.*

But if, he continues, the bowing be, not at a principal node, but near one, "the corresponding harmonic is extremely loud," a fact which has been very prettily verified by Krigar-Menzel,[3] who showed also that its

[1] For example, D. Bernoulli, in his "Memoir sur les sons produit par les tuyaux d'orgue," likens the action of a bow to that of a toothed wheel, and J. Antoine [Ann. de chim. et phys., Sér. III, **27**, 191; Pogg. Ann., **81**, 544 (1850)] speaks of "a series of gentle blows." Duhamel [C. R., **3**, 646, (1836); C. R., **9**, 567, (1839); Mem. d. Savants étrang., **8**, 131 (1843)] properly assumes the action to be frictional, but his treatment is not entirely satisfactory. That of F. Lippich [Mittheil. d. deut. math. Ges. in Prag, **1**, 118 (1892)] is much better, but it would scarcely have been possible apart from the experimental work which preceded it.

[2] Phil. Trans. of the Roy. Soc. of London, $\mathbf{90}_1$, 106 (1800).

[3] "Ueber d. Bewegung gestrich. Saiten," Inaug. Diss., Berlin, 1888. See also J. Ritz, "Zusammensetzung d. Klänge d. Streichinstr.," Inaug. Diss., München, 1883.

phase changes suddenly by π when the bowing point passes through the node.

A notable advance was made by Helmholtz,[4] who, with the aid of Lissajous's vibration microscope, proved

HELMHOLTZ'S LAW: *When a string is bowed at an aliquot point* $[1/k]$, *the part of the string immediately under the bow moves to and fro with constant velocities, whose ratio is equal to the ratio* $[1/(k-1)]$ *of the segments into which the string is divided by the point in question.*[5]

He also formulated the following

VELOCITY LAW: *The smaller of these two velocities has the same direction as that of the bow and is equal to it.*

In other words, "This point of the string adheres fast to the bow and partakes of its motion . . . then is torn off and jumps back to its first position . . . till the bow again gets hold of it." From these facts he derived his well known solution,

$$u = A \sum_n \frac{1}{n^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi t}{T},$$

where l is the length of the string, $2T$ is its period, and A is a constant depending on the velocity of the bow. This single expression satisfies the necessary conditions for any aliquot case, and may be called Helmholtz's major solution. In any particular case it must be modified, in accordance with Young's law, by the omission of the kth, $2k$th, etc., terms; that is, by subtracting the expression

$$u' = A \sum_m \frac{1}{k^2 m^2} \sin \frac{km\pi x}{l} \sin \frac{km\pi t}{T},$$

which may be called the minor solution for the case in hand. If the point bowed is near one end of the string, as is always true in musical practice, the minor solution is relatively unimportant and may usually be neglected altogether. This work of Helmholtz's solves the problem for

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[4] "On the motion of the strings of a violin," Proc. Glasgow Phil. Soc., Dec. 19, 1860; reprinted in Phil. Mag., Ser. 4. **21**, 393 (1861). See also Helmholtz, Sensations of Tone, etc., 3d Eng. ed., 80–88, 384–387; Rayleigh, Theory of Sound, 2d ed., 208 *et seq.*; Donkin, Acoustics, 2d ed., 136–144.

[5] An aliquot point is a point whose distance from one of the fixed ends is $1/k$th of the length of the string, where k is an integer.

any aliquot bowing point completely, and has been ingeniously verified by a number of observers.[6]

Non-aliquot, rational points were considered for the first time by Krigar-Menzel and Raps,[7] who showed that Helmholtz's law must, in the general case, be replaced by

KRIGAR-MENZEL'S LAW: *When a string is bowed at any rational point* p/q *where* p *and* q *are prime to each other, the part of the string immediately under the bow moves to and fro with constant velocities, whose ratio depends only on* q *and is* $1/(q-1)$.

This law, which includes Helmholtz's law as a special case, they verified for values of q up to 10, at which point it seemed to break down. In general, this breaking-point would be determined in some obscure fashion by such circumstances as the pressure and speed of the bowing, the material and size of the strings used, etc. It can easily be proved that in those cases in which Krigar-Menzel's law does hold, this law is enough, with Young's law and Helmholtz's velocity law, to determine the motion of the string uniquely. The theory is therefore complete for all simple rational cases, whether aliquot or not. It must be remembered, however, that it rests wholly on an empiric foundation, and that the foregoing laws, which are that foundation, were enunciated solely for the case of a string vibrating transversely under the action of a violin bow. This paper will show that they are equally applicable to the case of a string vibrating longitudinally under the action of a rough wheel.

The string upon which the following experiments were performed was a steel wire about 0.4 mm. in diameter (density 8.6; Young's modulus, as calculated from the observed frequency, 23,430 kg. wt. per mm^2). It was stretched horizontally, one end passing over a pulley and supporting 10 kgs., rather less than one third the breaking weight. Effective ends for the vibrating portion were obtained by clamping the wire firmly to the faces of two cast-iron blocks, each about 5 kg. in weight. The length of the vibrating portion was 676 cms. During most of the work a node was formed in the middle of the wire by means of a brass knife edge, the pressure on which was that due to a displacement of the wire of about a centimeter from its position of equilibrium. The wire was then rubbed on one side of the bridge and observed on the other, which

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[6] Mach, Pogg. Ann., **134**, 311 (1868); Clem. Neumann, Sitzber. I, and R. Acad. d. Wissenschaften, Wien, Math. Cl., 41_2, 89 (1870); Clifton, cited by Donkin, Acoustics, p. 138. See also Mach, Zeitsch. f. d. phys. u. chem. Unterr., **1**, p. 264 (1888); Fernbach, ibid., **9**, 239 (1896); Kuhfahl, ibid., **10**, 92 (1897).

[7] Wied. Ann., **44**, 623 (1891).

made all parts of a vibrating segment accessible, and furthermore kept the point which was being observed almost entirely free from the lateral motion which imperfections in the shape of the rubbing wheel would otherwise have caused.

The excitor or "bow" of these experiments was a chamois-covered wheel dusted with powdered rosin. In order to give an unbroken rubbing surface the chamois was cut from a skin in the form of a ring and stretched over the edge of the wheel, a method which gave very satisfactory results. During the first part of the work a wooden wheel 15.1 cms. in diameter was driven by a 1/12 H. P., A. C. motor through a system of wooden pulleys and belts which could be so adjusted as to give the peripheral velocity any one of eleven values from 5 to 91 cms. per second. A more satisfactory wheel 4.9 cms. in diameter was subsequently made of brass, to avoid warping, and was driven by the same motor through friction gears of brass and rubber so adjustable as to give a continuous variation in its peripheral velocity from about 3 cms. per second up; it was mounted on the slide rest of a small lathe bed, the two feed screws giving slow motions along and perpendicular to the wire.

These experimental conditions are unusually favorable, as compared with those of previous observers, in three respects. In the first place, as has already been noted, the use of a wheel instead of a bow enables a given state of vibration to be kept up indefinitely and to be reproduced at will, so that comparable observations can be made at different parts of the wire, and the corresponding "velocity of the bow" measured. In the second place, the vibrating segments are much longer than would have been possible for transverse vibrations, so that given points can be more accurately located and the effective width of the "bow" is more nearly negligible. And finally, the system is very nearly "perfectly flexible" because the sharp corners of a transversely vibrating string are replaced by points whose only peculiarity is a change in the law of distribution of density. It is very true that sudden jumps in this law necessarily imply instantaneous changes of finite magnitude in the velocities of the elements of the wire, but, other things being equal, "corners" would not be rounded off so much for longitudinal as for transverse vibrations. On the other hand, the amplitudes obtainable in this way, even with long strings, are much smaller than those of transverse vibrations, and furthermore it is always difficult to perform mechanically so delicate an operation as that of bowing a string, so that the more complicated vibration forms have sometimes been very hard to set up.

§ 2. Aliquot Points — Experimental Work.

The observations to be described in this section were made with the help of glass balls from 0.1 mm. to 0.2 mm. in diameter, which were formed by melting the end of a fine glass fibre in a flame, broken off, and fastened on the wire with a touch of shellac. When illuminated from the side by a Nernst filament behind a small hole in a screen, and observed through a stationary microscope, such a ball, acting as a spherical mirror, gives a bright point image of the source. When the wire is vibrating this point of light becomes a line, and, in the case of the rubbed string, this line is always bisected by the position of the point when the wire is at rest; half its length is, therefore, the amplitude of the vibration at the point in question. If the motion were transverse, the curve obtained by plotting the amplitudes which correspond to a single speed of the rubbing wheel, against distances along the wire, would be one

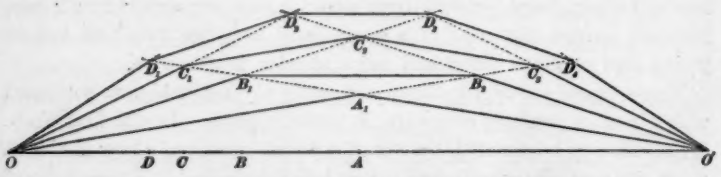

Figure 1. The envelopes for the aliquot cases $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

boundary of a region actually swept over by the wire during each vibration, that is, a region which one would see if a white string were vibrating over a black background. In other words, this curve would be the envelope of the various configurations of the wire. In the case of longitudinal vibrations these geometric notions are inapplicable, but it will still be convenient to plot amplitudes as though they were transverse, and to call the resulting curve the envelope of the corresponding vibration form. Such envelopes were obtained for each of the aliquot cases $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, and were found to consist of 2, 3, 4, and 5 straight lines respectively, the corners corresponding to points which divide the wire into 2, 3, 4, and 5 equal parts respectively. The shapes and relative magnitudes of these four envelopes for the same bowing speed are shown in Figure 1, the amplitude of the point bowed increasing directly as the distance of that point from the more distant end of the string,[8] and the other corners being determined by the intersections of two sets of rays

[8] In other words, the points A_1, B_1, C_1, and D_1 are collinear.

drawn from the ends of the base line in the manner indicated.[9] In the general case, $1/k$, the envelope would be k straight lines similarly determined. When the speed of the rubbing wheel was varied, all other conditions remaining the same, the amplitude was found to vary directly as the peripheral velocity of the wheel, a fact which was tested for a considerable range of velocities in each of the four cases considered. This fact is an obvious consequence of Helmholtz's two laws.

In all, 331 amplitudes were measured, each observation involving, in general, six settings of a filar micrometer. The results are summarized in Table I. The first column gives the position of the point rubbed, and

TABLE I.

Point rubbed.	Point observed.	Peripheral Velocity of Rubbing Wheel.	$M \times 10^4$, in cms.	No. of Observations.	Weight.
$\frac{1}{2}$	.091 to .895	22.4	$3.29 \pm .01$	62	6
	.497	5.1 to 90.9	$3.28 \pm .01$	62	6
$\frac{1}{3}$	.091 to .910	30.4	$3.27 \pm .01$	39	4
	3.37	9.9 to 59.3	$3.19 \pm .01$	7	1
$\frac{1}{4}$	.050 to .922	25.7	$3.20 \pm .01$	42	4
	.500 and .750	5.1 to 59.3	$3.24 \pm .02$	40	4
$\frac{1}{5}$	.400 to .919	18.0	$3.34 \pm .01$	67	7
	.400	9.9 to 25.7	$3.24 \pm .01$	12	1

the second, the position or range of positions of the point observed, in fractions of the length of a vibrating segment; the third gives the peripheral velocity of the rubbing wheel in cms. per second. In the fourth column is given a mean standard amplitude M, obtained from the actually observed amplitudes m, first, by multiplying each of them by an appropriate factor obtained directly from the geometry of Figure 1, giving a

[9] The distance A A_1 having been arbitrarily assumed, the points B_1, C_1, and D_1 on the line O′ A_1 are determined by the bowing points B, C, and D, and themselves determine the rays through O. The symmetry of the figure determines the corresponding rays through O′.

value for the fundamental ordinate AA_1 of that figure, and second, by dividing this fundamental ordinate by the corresponding peripheral velocity of the rubbing wheel. The resulting M is, therefore, the amplitude in centimeters, of the motion which the middle point of the string would have if the string were rubbed at that middle point with a speed of one centimeter per second. It should be a constant under all circumstances. The smallness of the probable error in the second, fourth, sixth, and eighth lines of the table is a verification of the speed law stated in the last paragraph. Its smallness in the other four cases shows that the shape of each individual envelope is that which Figure 1 predicts. This is shown in another way in Figure 2, in which the observations of these groups are plotted in the manner already mentioned. The envelope in each figure is the best line *of the given shape* that could be drawn. And finally the constancy of M throughout the table shows that the relative magnitude of the envelopes is as described. This verification is not quite so good as the individual probable errors would lead one to expect, because of certain errors, largely in the adjustment of the brass bridge, which are constant throughout any one set of observations, but vary from set to set. The weighted mean of the results in the table is

$$M = .000327 \pm .000001 \text{ cms.}$$

In the case $\frac{1}{2}$ to which this standard ordinate would correspond, the two velocities under the bow would be equal, if Helmholtz's law is applicable in longitudinal cases. If F be the frequency of the string, $4MF$ would be the distance travelled by its middle point in one second, and this should, according to the velocity law, be equal to the corresponding velocity of the bow, that is, to 1 cm. per second. Therefore $F = 1/4M$. This formula gives

$$F \text{ (calculated)} = 764 \pm 2.$$

A Valentine and Carr set of tonometer forks gave, by the method of beats,

$$F \text{ (observed)} = 762,$$

an excellent quantitative verification of Helmholtz's two laws for longitudinal vibrations.

§ 3. Aliquot Points — Theory.

In the preceding section it has been shown that the amplitudes which have been measured in the various aliquot cases considered could all be

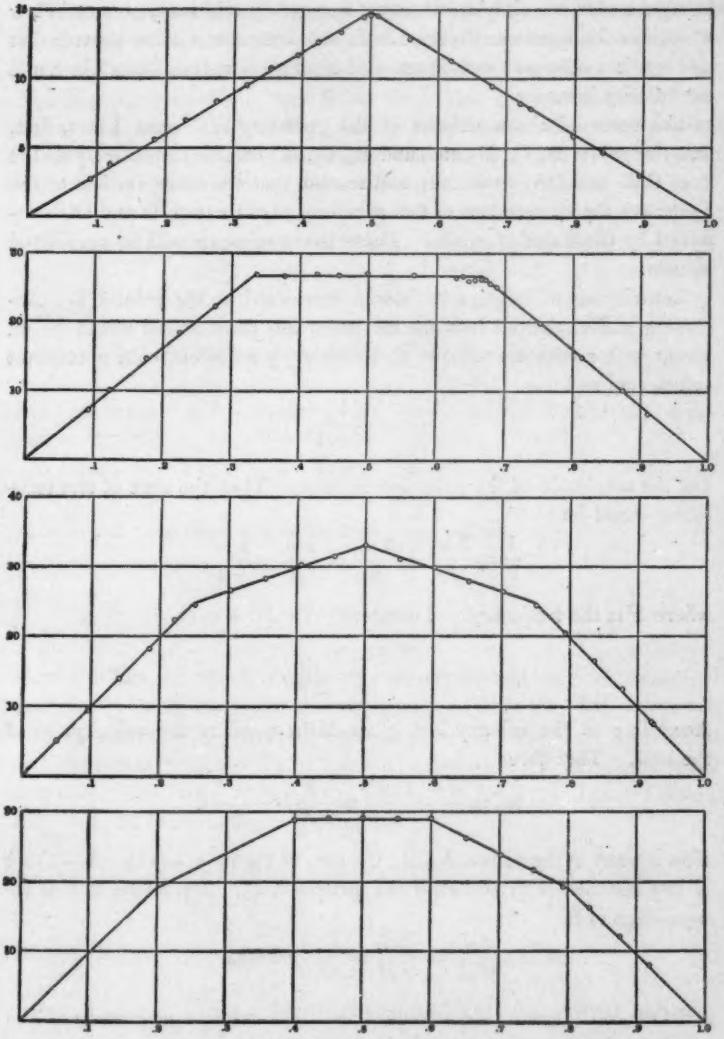

FIGURE 2. Amplitude measurements on a string bowed at points $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, and $\frac{1}{9}$ of its length from one end. The ordinates are the amplitudes actually observed, in hundredths of a millimeter. The node formed by the brass ridge is at the right.

brought under one simple, but as yet apparently arbitrary, geometric law which has been called Figure 1. In this section it will be proved that this law is a necessary consequence of Young's law, Helmholtz's law, and the velocity law.

The essential characteristics of the geometry of Figure 1 are, first, that the points B_1, C_1, D_1, etc., and B_2, C_2, D_4, etc., lie respectively on the lines $O'A_1$ and OA_1 produced; and second, that the other vertices of the figure are the intersections of two pencils of rays through O and O′ determined by these sets of points. These two statements will be considered in turn.

Let a string of length l be bowed downward at the point $1/k$. According to Helmholtz's law, the motion of the point bowed would be an ascent with a constant velocity f_0, followed by a descent with a constant velocity g_0, and

$$\frac{g_0}{f_0} = \frac{1}{k-1}.$$

Let the amplitude of the point bowed be m_0. Then the time of one oscillation would be

$$\frac{1}{F} = \frac{2\,m_0}{f_0} + \frac{2\,m_0}{g_0} = \frac{2\,m_0}{g_0}\,\frac{k}{k-1},$$

where F is the frequency. Therefore

$$m_0 = \frac{k-1}{k}\,\frac{g_0}{2\,F}.$$

According to the velocity law, g_0 would be equal to the velocity, V, of the bow. Therefore

$$m_0 = \frac{k-1}{k}\,\frac{V}{2\,F} = \frac{V}{2\,Fl}\cdot\frac{k-1}{k}\,l.$$

Now for any of the points, A_1, B_1, C_1, etc., of Figure 1, say C_1, $(k-1)\,l/k$ is the distance O′C, and m_0 is the distance CC_1. Therefore, if V is independent of k,

$$\frac{AA_1}{O'A} = \frac{BB_1}{O'B} = \frac{CC_1}{O'C} = \text{etc.},$$

and $O'A_1B_1C_1D_1$, etc., is a straight line. The relative magnitude of the envelopes is, therefore, that which the figure indicates.

There remains to be proved only the fact that the shape of the kth envelope (a) as given by Figure 1, and (b) as calculated theoretically, is k straight lines inscribed in a parabola.

Take the line A O′ of Figure 1 as an x axis and a line perpendicular to it at A as a y axis, and let the units of length along the two axes be A O′ and A A_1 respectively. Call the vertices of the kth envelope O or K_0, K_1, K_2, . . . K_{k-1}, and K_k or O′. Let the coördinates of any vertex, K_j, be $x(j, k)$ and $y(j, k)$. When $j = 1$ the corresponding vertex lies on the line O′A_1, whose equation is

$$y = -x + 1.$$

Also

$$x(1, k) = -1 + \frac{1}{k} \cdot 2 = \frac{2-k}{k}.$$

Therefore

$$y(1, k) = -x(1, k) + 1 = -\frac{2-k}{k} + 1 = 2\frac{k-1}{k}.$$

Similarly when $j = k - 1$, the vertex lies on the line O A_1, whose equation is

$$y = x + 1.$$

Also

$$x(k-1, k) = -1 + \frac{k-1}{k} \cdot 2 = \frac{k-2}{k}.$$

Therefore

$$y(k-1, k) = x(k-1, k) + 1 = \frac{k-2}{k} + 1 = 2\frac{k-1}{k}.$$

Now the law by which Figure 1 was constructed may be expressed analytically by the statement that the point $[x(j, k), y(j, k)]$ is collinear with the points $[x(j-1, k-1), y(j-1, k-1)]$, and $[-1, 0]$, and also with the points $[x(j, k-1), y(j, k-1)]$, and $[+1, 0]$. It follows, in the first place, that $[x(j, k), y(j, k)]$, $[x(1, k-j+1), y(1, k-j+1)]$, and $[-1, 0]$ are collinear. Therefore

$$\frac{y(j, k)}{y(1, k-j+1)} = \frac{x(j, k) + 1}{x(1, k-j+1) + 1}.$$

Substituting the values found above and simplifying gives

$$(k-j) \cdot x(j, k) - y(j, k) = j - k.$$

Similarly $[x(j, k), y(j, k)]$, $[x(j, j+1), y(j, j+1)]$, and $[1, 0]$ are collinear. The resulting equation is

$$j \cdot x(j, k) + y(j, k) = j.$$

These two equations, when solved for $x(j, k)$ and $y(j, k)$ give

$$x(j, k) = \frac{2j - k}{k},$$ [10]

$$y(j, k) = \frac{2j}{k}(k - j).$$

Now the symmetry of Figure 1 shows that the equation of the desired parabola, if one exists, must be of the form

$$x^2 = 2p_k(a_k - y),$$

where p_k and a_k are undetermined parameters corresponding to the kth envelope. Pass such a parabola through the vertex $[-1, 0]$ and any other vertex $[x(j, k), y(j, k)]$ of this envelope. The equations for determining p_k and a_k are

$$1 = 2p_k a_k$$

and

$$\frac{(2j - k)^2}{k^2} = 2p_k\left[a_k - \frac{2j}{k}(k - j)\right],$$

and these give

$$p_k = \frac{1}{k} \quad \text{and} \quad a_k = \frac{k}{2},$$

each of which is independent of j. Therefore the parabola

$$x^2 = \frac{2}{k}\left(\frac{k}{2} - y\right)$$

passes through all the vertices of the kth envelope, and the first part of the proof is completed.

In the second part of the proof[11] it will be convenient to transfer the origin to the left-hand end of the string. The length of the string is l, as before. Let it be bowed downward at the point $x_0 = l/k$. Helmholtz's solution may be written

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[10] This equation can be put in the form $x(j,k) = -1 + \frac{j}{k}\cdot 2$, which proves that the projections of the vertices of the kth envelope are points which divide the length of the string into k equal parts.

[11] This part of the proof follows closely the method suggested in Helmholtz's paper of 1860.

$$y = u - u' = \frac{8H}{\pi^2} \sum_n \frac{1}{n^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi t}{T}$$

$$- \frac{8H}{\pi^2} \sum_m \frac{1}{k^2 m^2} \sin \frac{km\pi x}{l} \sin \frac{km\pi t}{T},$$

where H is a constant. If the first term were the complete solution, the motion of any point x of the string would be an ascent from h units below equilibrium to h units above it, with a constant velocity f, during an interval of length 2τ, followed by a return to the original position, with another velocity g, during an interval of length $2(T-\tau)$, where

$$h = 4H \frac{x}{l} \frac{l-x}{l},$$

$$f = \frac{4H}{T} \frac{l-x}{l},$$

$$g = \frac{4H}{T} \frac{x}{l},$$

$$\tau = T \frac{x}{l}.$$

Suppose any one point to be going up through its position of equilibrium at the time $t = 0$. Then all points would go up through their positions of equilibrium at the times 0, $2T$, $4T$, etc., and would go down through those positions at the times T, $3T$, $5T$, etc.; and each of these times would be for each point the middle of an interval, during which its velocity is constant. The first of the above equations shows the envelope of this major vibration to be a parabola, whose vertex is at the point $(l/2, H)$.[12]

The case in hand is modified by the presence of a term

$$u' = \frac{8H}{\pi^2} \sum_m \frac{1}{k^2 m^2} \sin \frac{km\pi x}{l} \sin \frac{km\pi t}{T}$$

$$= \frac{8}{\pi^2} \frac{H}{k^2} \sum_m \frac{1}{m^2} \sin \frac{m\pi x}{l/k} \sin \frac{m\pi t}{T/k}.$$

The motion corresponding to this term alone would be exactly similar to that described above, except that the constants l, T, and H would be

[12] For a proof of these facts see any of the references mentioned in the note on page 695.

replaced by $l' = l/k$, $T' = T/k$, $H' = H/k^2$. That is, at the time $t = 0$, each point in the first kth of the string would be ascending through its position of equilibrium with a velocity

$$f_1' = \frac{4\,H'}{T'}\,\frac{l' - x}{l'} = \frac{4\,H}{T}\left(\frac{1}{k} - \frac{x}{l}\right),$$

and this motion would continue until the time

$$t = \tau_1' = T'\,\frac{x}{l'} = T\,\frac{x}{l} = \tau,$$

the turning point coinciding with that of the major vibration. It would then descend during an interval centred at $t = T/k$ with a velocity

$$g_1' = \frac{4\,H'}{T'}\,\frac{x}{l'} = \frac{4\,H\,x}{T\,l},$$

then ascend during an interval centred at $t = 2\,T/k$ with the velocity f_1' as before, and so on indefinitely.

The complete motion, in the actual case, of any point x, $0 < x < l/k$, is, therefore, an ascent from its position of equilibrium, during the interval $0 < t < \tau$, with a velocity

$$f - f_1' = \frac{4\,H}{T}\,\frac{l - x}{l} - \frac{4\,H}{T}\left(\frac{1}{k} - \frac{x}{l}\right) = \frac{4\,H}{T}\,\frac{k - 1}{k};$$

followed by a number of descents, the first of them during an interval centred at $t = T/k$, with a velocity

$$g - g_1' = 0,$$

the second during an interval centred at $t = 2\,T/k$, with a velocity

$$g + f_1' = \frac{4\,H}{T}\,\frac{x}{l} + \frac{4\,H}{T}\left(\frac{1}{k} - \frac{x}{l}\right) = \frac{4\,H}{T}\,\frac{1}{k},$$

the third during an interval centred at $t = 3\,T/k$, with a velocity 0 as before, and so on. At the time $t = T$, the point is descending through equilibrium with a velocity which is either $4H/kT$ or 0, according as k is even or odd. The other half of a complete vibration is most easily determined by the fact that both components are odd functions of the time. The whole curve $y = \phi\,(t)$ for any point in the first kth of the string is, therefore, like a flight of steps with one steep ascent and $k - 1$

less steep descents connecting k horizontal stretches. A similar proposition is true for points in the last kth of the string.[13]

The portion of the envelope between $x = 0$ and $x = l/k$ is given by

$$m = \left(\frac{4Hk-1}{T}\frac{}{k}\right)\left(T\frac{x}{l}\right) = 4H\frac{k-1}{k}\frac{x}{l},$$

and is a straight line intersecting the parabola

$$h = 4\,\mathrm{H}\,\frac{x}{l}\frac{l-x}{l}$$

at the points $(0, 0)$ and $(l/k,\ 4H\,(k-1)/k^2)$, which agrees with what was to be proved.

In the second kth of the string, a point under the influence of the minor vibration alone, would, at the time $t=0$, be descending through equilibrium with a velocity

$$g_2' = \frac{4H'}{T'}\frac{x-l'}{l'} = \frac{4H}{T}\left(\frac{x}{l}-\frac{1}{k}\right),$$

and this descent would last until the time

$$t = T' - \tau_2 = T' - T'\frac{x-l'}{l'} = T\left(\frac{2}{k}-\frac{x}{l}\right).$$

It would be followed by an ascent during an interval centred at $t = T/k$ with a velocity

$$f_2' = \frac{4H'}{T'}\frac{l'-(x-l')}{l'} = \frac{4H}{T}\left(\frac{2}{k}-\frac{x}{l}\right).$$

The whole duration of this interval would be

$$2\tau_2 = 2T\left(\frac{x}{l}-\frac{1}{k}\right),$$

---

[13] This work shows that the figure in Helmholtz's 1860 paper is incorrect. It is true that the differential equation in question has a solution corresponding to the function $y = f(t)$, which his figure represents, and that this solution satisfies the boundary conditions and both of Helmholtz's laws, as can easily be proved by the method of section 5 of this paper. But it does not satisfy Young's law, the 6th, 18th, 30th, etc., partials being present, and it could not, so far as our present knowledge of the violin string goes, be actually set up by bowing a string at the point 1/6.

For some theorems which are very useful in this connection, see Wedmore, Journ. Inst. of El. Engg., **25**, 224 (1896); and Lyle, Phil. Mag., **6**, 549 (1903); ibid., **11**, 25 (1906).

and it would end at the time

$$t = T' + \tau_2 = T\frac{x}{l} = \tau,$$

so that one of the turning points of the minor vibration coincides with that of the major vibration, as before. This is true in all cases.

The complete motion of a point in the second kth of the string is, therefore, (a) two ascents, the first during the interval

$$0 < t < T\left(\frac{2}{k} - \frac{x}{l}\right),$$

with a velocity

$$f + g_2' = \frac{4H}{T}\frac{l-x}{l} + \frac{4H}{T}\left(\frac{x}{l} - \frac{1}{k}\right) = \frac{4H}{T}\frac{k-1}{k};$$

and the second during the interval

$$T\left(\frac{2}{k} - \frac{x}{l}\right) < t < T\frac{x}{l},$$

with a velocity

$$f - f_2' = \frac{4H}{T}\frac{l-x}{l} - \frac{4H}{T}\left(\frac{2}{k} - \frac{x}{l}\right) = \frac{4H}{T}\frac{k-2}{k};$$

followed by (b) a number of descents, the first during an interval centred at $2T/k$, with a velocity

$$g - g_2' = \frac{4H}{T}\frac{x}{l} - \frac{4H}{T}\left(\frac{x}{l} - \frac{1}{k}\right) = \frac{4H}{T}\frac{1}{k};$$

the second during an interval centred at $3\,T/k$, with a velocity

$$g + f_2' = \frac{4H}{T}\frac{x}{l} + \frac{4H}{T}\left(\frac{2}{k} - \frac{x}{l}\right) = \frac{4H}{T}\frac{2}{k};$$

the third during an interval centred at $4T/k$ with the velocity $4H/Tk$ as before, and so on. In no case will the curve $y = \phi(t)$ for any point not in the first or last kth of the string have any horizontal portions whatever, but both the ascending and the descending portions will be somewhat "crumpled."

These facts are very prettily illustrated by the appearance under a microscope of the bright lines upon which the observations of the last section were made. When the observed point is in one of the end $k$ths

of the wire there is always a very distinct spot of light at each end of the line, and $k-2$ other spots equally spaced between them, corresponding to the k horizontal portions of the vibration curve; but in all other cases the line is of nearly uniform intensity, even at the ends.

The portion of the envelope between $x = l/k$ and $x = 2\,l/k$ is given by

$$m = \frac{4Hk-1}{T}\frac{}{k} \cdot T\left(\frac{2}{k} - \frac{x}{l}\right) + \frac{4Hk-2}{T}\frac{}{k} \cdot 2\,T\left(\frac{x}{l} - \frac{1}{k}\right)$$
$$= 4H\left(\frac{2}{k^2} + \frac{k-3}{k}\frac{x}{l}\right),$$

and is a straight line intersecting the parabola

$$h = 4H\frac{x}{l}\frac{l-x}{l}$$

at the points $(l/k,\ 4H(k-1)/k^2)$ and $(2\,l/k,\ 8H(k-2)/k^2)$, as was to be proved.

Similar reasoning can be applied to a point in the third kth, or, in general, to any point x, where

$$\frac{j\,l}{k} < x < \frac{(j+1)\,l}{k}, \text{ and } j < k;$$

but it is necessary to distinguish between a case in which j is even and one in which it is odd. The resulting formulae are given in Table II. In each case the envelope is a straight line intersecting the above mentioned parabola at the points $[j\,l/k,\ 4Hj\,(k-j)/k^2]$ and $[(j+1)\,l/k,\ 4H(j+1)(k-j-1)/k^2]$, and the second part of the proof is completed.

The geometry of Figure 1, which was obtained experimentally for longitudinal vibrations, is, therefore, a verification of the established theory for the transverse vibrations of a string bowed at an aliquot point.

It is easy to construct from the data of Table II either the curve $y = \phi(t)$ for a given x, or the curve $y = \psi(x)$ for a given t, in any desired aliquot case. The results for $k = 2$, 3, and 4 are given in Figures 3, 4, and 5. From them one can get a reasonably definite idea of the general case. It is interesting to notice that the vibration of a string bowed in the middle is identical with that of a string plucked in the middle, although under other circumstances the corresponding modes of vibration of a bowed and of a plucked string are wholly different.

TABLE II.

$\frac{jl}{k} \leq x \leq \frac{(j+1)l}{k}, j<k.$		
	When j is even.	When j is odd.
First ascending velocity	$\frac{4H}{T}\frac{k-(j+1)}{k}$	$\frac{4H}{T}\frac{k-j}{k}$
Second ascending velocity	$\frac{4H}{T}\frac{k-j}{k}$	$\frac{4H}{T}\frac{k-(j+1)}{k}$
Number of ascents	$j+1$	$j+1$
First descending velocity	$\frac{4H}{T}\frac{j}{k}$	$\frac{4H}{T}\frac{j}{k}$
Second descending velocity	$\frac{4H}{T}\frac{j+1}{k}$	$\frac{4H}{T}\frac{j+1}{k}$
Number of descents	$k-j$	$k-j$
Duration of first interval	$T\left[\frac{x}{l}-\frac{j}{k}\right]$	$T\left[\frac{j+1}{k}-\frac{x}{l}\right]$
Duration of second interval	$2T\left[\frac{j+1}{k}-\frac{x}{l}\right]$	$2T\left[\frac{x}{l}-\frac{j}{k}\right]$
Duration of third interval	$2T\left[\frac{x}{l}-\frac{j}{k}\right]$	$2T\left[\frac{j+1}{k}-\frac{x}{l}\right]$
Envelope	$\frac{4H}{k}\left\{\frac{x}{l}[k-2j-1]+\frac{j(j+1)}{k}\right\}$	

Showing the motion of any point, x, of a string bowed at the aliquot point $1/k$. The table describes a half vibration, from equilibrium ascending to equilibrium descending. The ascending velocities alternate until $(j+1)$ ascents are accomplished; then the descending velocities alternate until equilibrium is reached; then the vibration is continued below equilibrium in such a way as to make the displacement an odd function of the time. The last ascent is always with the smaller of the ascending velocities; similarly the first descent is with the smaller of the descending velocities. The table describes $(k+1)$ straight lines, but the first and the last of them are halves of longer stretches in the complete curve which thus consists essentially of $2k$ straight lines. This shows why the duration of the first velocity is given in the table as half that of the third; the other half of the interval is thought of as belonging to the preceding half-vibration.

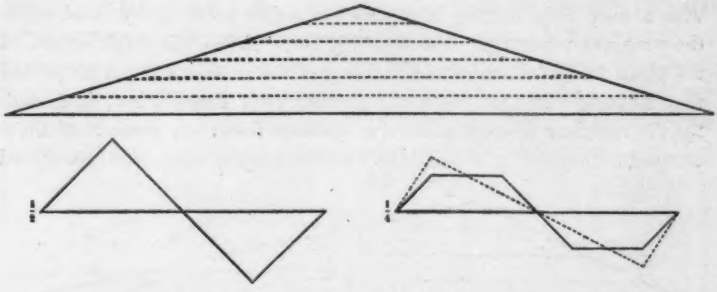


FIGURE 3. The case $\frac{1}{4}$. The upper figure shows the configurations of the string at the times 0, $T/12$, $2T/12$, . . . $6T/12$, $2T$ being the period; the string returns to equilibrium through the same forms; in this, and in the following cases, the configurations during the second half of a complete vibration can be obtained by turning the page upside down. The lower figures show the curves $u = \phi(t)$ for the points $\frac{1}{2}$ and $\frac{1}{4}$. The dotted lines in this, and in the similar figures to follow, show the motion due to the major vibration alone. It is interesting to notice that this mode of vibration is identical with that of a string plucked at its middle point.

§ 4. RATIONAL POINTS — KRIGAR-MENZEL'S LAW.

The experimental method described in section 2 has two disadvantages. In the first place, it is not easy, especially in a complex case, to discover any vibration form that corresponds to a given envelope; and in the second place, an envelope is sometimes, and perhaps always, not enough to determine its vibration form uniquely. For example, in the case of the struck string, if the initial conditions be taken in the form

$$u = 0 \text{ when } t = 0,$$

$$\frac{\partial u}{\partial t} = 0 \text{ when } t = 0, \quad \text{and} \quad 0 < x < x_0,$$

$$\frac{\partial u}{\partial t} = C \text{ when } t = 0, \quad \text{and} \quad x_0 < x < x_0 + a,$$

$$\frac{\partial u}{\partial t} = 0 \text{ when } t = 0, \quad \text{and} \quad x_0 + a < x < l,$$

it can easily be shown that the envelope depends only on the constants a and C and not at all upon x_0. A second standard method of observation was therefore made use of, and it proved to be more powerful, although much less accurate, than the first. The point of a fine needle was broken off and fastened to the wire with shellac,[14] and a microscope slide

[14] The added weight was that of about half a millimeter of wire.

with a very light coating of smoke was drawn under it by hand while the wire was vibrating. The resulting curve shows the displacement of the given part of the wire as a function of the time, and it can be proved that a single curve of this kind, together with Young's law, is enough for the complete determination of a vibration form. A number of these curves, corresponding to the three simplest aliquot cases, are reproduced

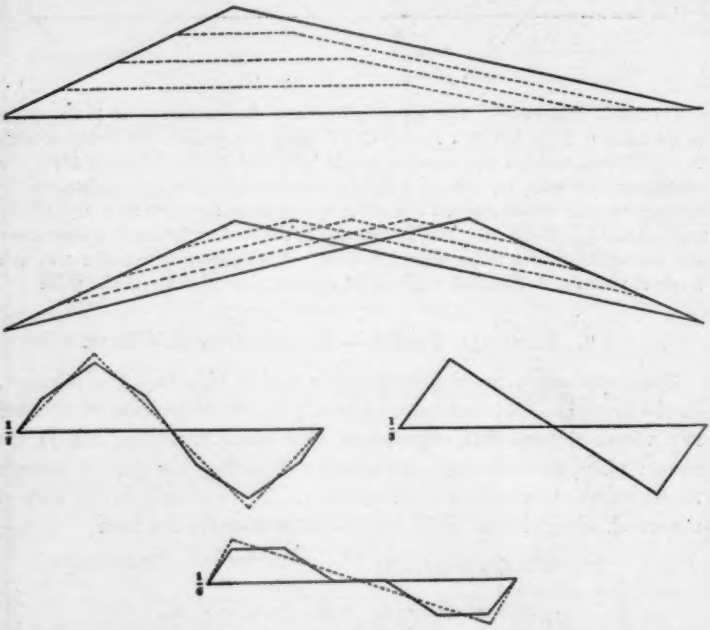


FIGURE 4. The case $\frac{1}{1}$. The upper figures show the configurations of the string at intervals of $T/12$, the first from 0 to $T/3$, and the second from $T/3$ to $2T/3$; the return to equilibrium is through forms symmetric to those of the upper figure. The lower figures show the motion of the points $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$.

photographically, much enlarged, in Figure 6, and their resemblance to the calculated curves of Figures 3, 4, and 5 is another and a final verification of the conclusions of the last two sections.

This section will describe a number of observations which have been made by this method to test Krigar-Menzel's law for the simpler rational cases. It will be remembered that, according to this law, the motion of the point bowed in the rational case p/q, is an ascent with a constant

velocity, followed by a descent with another constant velocity, the ratio of the lesser velocity to the greater being $1/(q-1)$. It follows that the duration of the greater velocity is $1/q$th of a complete period, or, speaking geometrically, that the length of the projection on the t axis of the steeper of the two straight lines which make up a typical segment of the curve $y=\phi(t)$ for the point in question, is $1/q$th of the length of that

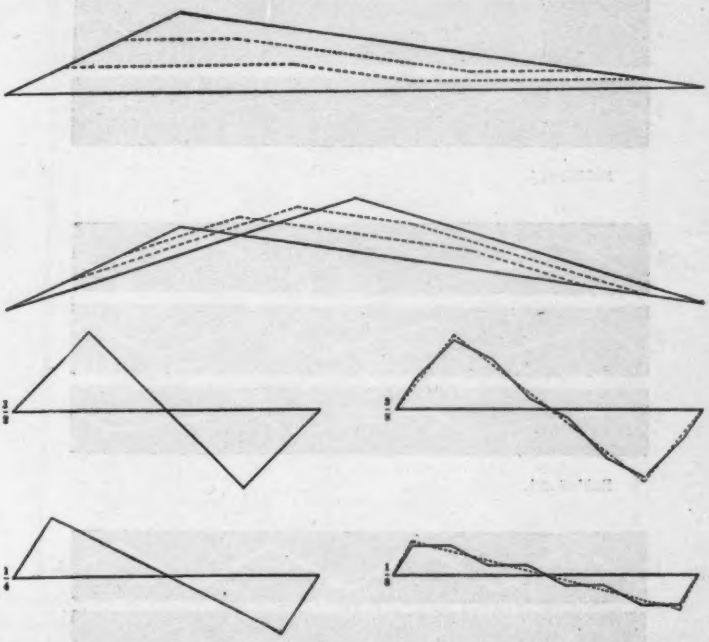
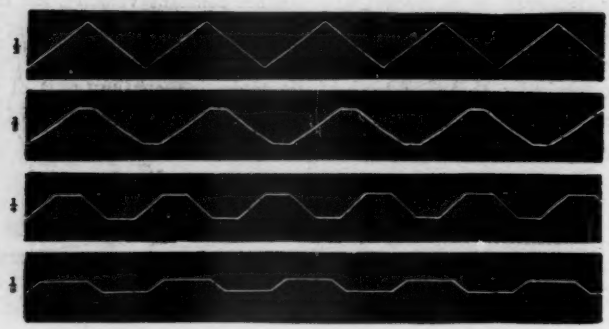


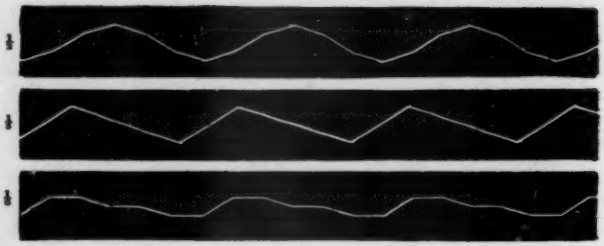
FIGURE 5. The case $\frac{1}{4}$. The upper figures show configurations at intervals of $T/12$, the first from 0 to $T/4$, and the second from $T/4$ to $T/2$; the return to equilibrium is through forms symmetric to those of the ascent. The lower figures show the motion of the points $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$.

segment. A number of measurements of this ratio for the various rational cases up to and including $q=10$ are presented in Table III.

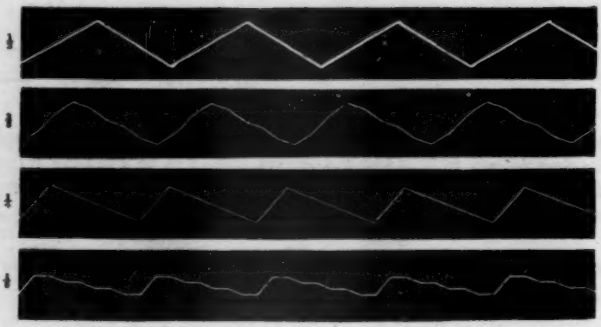
The observations summarized in this table were made with a Gaertner comparator whose stage was driven by a half-millimeter screw with a head reading to thousandths of a millimeter. Each slide was oriented so that the curve to be measured lay along the axis of the micrometer screw and readings were made of the settings necessary to bring



Rubbed at $\frac{1}{2}$.



Rubbed at $\frac{1}{3}$.



Rubbed at $\frac{1}{4}$.

FIGURE 6. Experimental u t curves for the simplest aliquot cases. (Cf. figures 3, 4, and 5.)

TABLE III.

Case.	Ratio predicted.	Ratio found.	Per Cent Difference.
$\frac{1}{2}$	.500	.503	+0.6
$\frac{1}{3}$	.333	.332	−0.3
$\frac{1}{4}$	.250	.247	−1.2
$\frac{1}{5}$	.200	.201	+0.5
$\frac{2}{5}$	.200	.202	+1.0
$\frac{1}{6}$	.167	.166	−0.6
$\frac{1}{7}$	.143	.142	−0.7
$\frac{2}{7}$	.143	.141	−1.4
$\frac{3}{7}$	.143	.144	+0.7
$\frac{1}{8}$	.125	.126	+0.8
$\frac{3}{8}$	.125	.123	−1.6
$\frac{1}{9}$	.111	.112	+0.9
$\frac{2}{9}$	.111	.109	−1.8
$\frac{4}{9}$	.111	.112	+0.9
$\frac{1}{10}$	.100	.100	0.0
$\frac{3}{10}$	.100	.100	0.0

Observations to test Krigar-Menzel's law. It is only fair to state that each of the ratios in this table is the best of a number of determinations. The ratio selected in each case does, however, take into account all of the 105 settings which were made on the slide in question.

twenty-one successive vertices under a vertical cross-hair. These gave the lengths of the projections on the t axis of ten consecutive ascents and of the ten descents immediately following them. Since the velocity of the slide when the tracing was made was not in general uniform, the averages of the values thus obtained had to be slightly modified to allow for the distortion of the time scale of the slide. The necessary correction, which was always small, was obtained from the values themselves on the supposition that the acceleration of the slide was uniform during the interval in question — about the fortieth part

of a second. Five such measurements were made on different parts of at least three different tracings for each case discussed, so that each of the observed ratios in Table III depends on fifty measurements of each of the quantities involved. This labor seemed to be necessary because, under high magnification, the tracing itself was so wide as to make the setting corresponding to a given vertex somewhat uncertain, and also because it was hoped that this averaging would eliminate the effect of local variations in the time scale of the slide due to any transverse or torsional vibrations that might be present. As a matter of fact such vibrations caused surprisingly little trouble even when the bridge was not used.

Three other sources of error were constantly in evidence. In the first place, it is assumed above that the vertical cross-hair is perpendicular to the axis of the screw, and an error of one degree in this adjustment might, under unfavorable circumstances, cause an error of more than two per cent in the observed ratio.[15] Eighteen sets of observations on photographic reductions of a carefully prepared drawing showed that the angle of error was actually about 40′ and its effect has been allowed for when necessary. A second error is involved in the orientation of the slide itself with respect to the axis of the screw. This was harder to allow for, but the adjustment could be made with considerable accuracy, and furthermore the resulting error would be somewhat diminished when the results for five tracings were averaged. And finally, it was difficult, with the comparatively crude apparatus which was used, to be sure that the slide was moving perpendicularly to the motion of the wire when the tracing was made; but in getting slides for measurement, this adjustment was repeated for each line, so as to take advantage of the averaging process already mentioned. Fortunately the resulting error in all three cases varies directly as the ratio of the amplitude of a tracing to the apparent length of its period, and in all cases this ratio was made very small by increasing the speed of the slide when the tracings were made.

Under the circumstances the agreement of the observed with the predicted values is reasonably satisfactory, the average departure being 0.8%, or, if account be taken of sine, −0.1%, while the average of the probable errors of the individual determinations is 0.5%. It is evident that Kriger-Menzel's law holds for the longitudinal vibrations

[15] One advantage of the method of section 2 is that it involves the versed sine of the corresponding angle of error rather than its sine.

considered in this paper as well as for the transverse vibrations of a bowed string.

§ 5. Integral Surfaces.

It is usual to solve problems connected with the vibration of bowed strings by Helmholz's method and to express the results as trigonometric series in x and t. A solution for the case $2/5$ can easily be obtained in this way, and is

$$u = \frac{5VT}{\pi^2} \sum_n \frac{1}{n^2} \frac{1}{\cos \frac{n\pi}{5}} \sin \frac{n\pi x}{l} \sin \frac{n\pi t}{T}, \qquad n \neq 5, 10, \text{ etc.},$$

where V is the velocity of the bow — a form which is entirely satisfactory for the theory of music, where the important thing is to know the intensities of the various harmonics produced, but one which gives very little information of any other kind. In the case of an aliquot point it is possible, with considerable labor, to work out from the trigonometric series a complete description of the configurations and motion of the string itself, because the various series in x or t which present themselves are easily recognizable standard forms;[16] but even in this, the simplest of all the non-aliquot cases, such a fortuitous proceeding would be very difficult, if not impossible,[17] and the only other obvious method of handling this form of the solution, namely by direct computation, is impracticable. The desired information can, however, be obtained by a simple graphical process, which it is the purpose of this section to describe.

An integral surface is a three-dimensional graph which represents the displacement u of any portion of the string at any time, as a function of x and t. Such surfaces can be handled in two dimensions by plotting their contour lines, $u =$ a constant, on the basal $x\,t$ plane, just as would be done on an ordinary map. For a string of length l, one is concerned only with that part of the $x\,t$ plane which lies between the lines $x = 0$ and $x = l$, and the displacement, u, is zero along each of these lines. Also, since the motion of such a string is necessarily periodic with a period $2T$, it is necessary to cover only so much of the above mentioned strip as lies between the lines $t = 0$ and $t = 2T$. If l and T are taken

[16] See section 3 of this paper.

[17] At least without the assistance of a complete set of experimentally determined $y\,t$ curves.

as the units of length and of time respectively, the differential equation becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

and the fundamental rectangle is two adjacent unit squares.

Such surfaces have been plotted from the data of Table II, for the aliquot cases, from $\frac{1}{2}$ to $\frac{1}{7}$, and are shown in the top line of the first of the accompanying plates. They have a number of common characteristics, and, in particular:

1. The displacement, u, is zero not only along the lines $x = 0$ and $x = 1$, but along the lines $t = 0$, $t = 1$, and $t = 2$ as well.

2. Each surface is made up entirely of plane facets, whose projections on the basal $x\,t$ plane are the squares formed by two sets of equidistant lines, parallel to the diagonals of the large squares, and determined as follows: one line of each set passes through A, the point of discontinuity in the known section, $u = \Phi(t)$, under the bow (see Figure 7); if the points where these lines intersect the line $t = 0$ be called B and C, B determines a parallel adjacent to AC, and C one adjacent to AB.

Conversely, if these two properties be assumed, it is possible to reconstruct the whole surface from the known section $u = \Phi(t)$. For the set of squares can be laid out in accordance with the second of the above rules; then the heights of the various corners on some arbitrary scale can be determined as follows (see Figure 7). Assume the height of A to be 4. Then D must be 2, for the known section drops uniformly from A to E. Since A is 4 above C, F must be 4 above G, or $0 + 4 = 4$.[18] Then since F is 4 above H, I must be 4 above D, or $2 + 4 = 6$. And I with A and C determines J. Furthermore, the heights of the three corners still undetermined must be equal to those of D, I, and J respectively, since E = K. The intermediate heights along the sides of the small squares and the contours within them can now be filled in without difficulty.

Fortunately, this same process, when carried through for any rational case that obeys Krigar-Menzel's law, gives the desired solution graphically. To show how it works in the general case, the construction of the surface for $\frac{2}{3}$ will be described in some detail with the help of

[18] The drop along one edge of an oblique section of a square prism must be equal to that along the edge opposite, and therefore to that along the opposite side of an adjacent prism, etc.

Figure 8. The parallels are laid out as before by means of the points A, B, and C; the height of A is taken arbitrarily as 4; and Krigar-Menzel's law gives D = 2. Then since G = 0, F = 0 + 4 = 4. Also F and H give I = J + 4 = 4, and E and I give K = D − 4 = − 2 and M = L + 4 = 4, which brings us back *via* N to the starting point A. One such round can always be made, and it will always cross the known section A E in such a way as to start a second round. In this case either of the pairs D I or D M enables one to make this second round and to complete the figure. In a more complicated case a third round, a fourth round, etc., would follow; and as new points are filled in, former

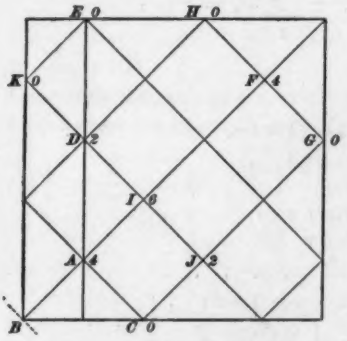


FIGURE 7. Showing the construction of the integral surface for the aliquot case ½.

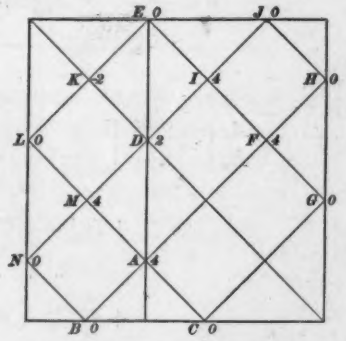


FIGURE 8. Showing the construction of the integral surface for the non-aliquot case ⅓.

rounds may be repeated, until the surface is completed. The method will always yield one, and only one, surface.

No attempt will be made in this paper to deduce the two rules mathematically from general principles.[19] As is so often necessary in the theory of La Place's equation, they are to be regarded simply as an arbitrary device for producing something that can subsequently be proved to be (*a*) a solution, and (*b*) the only possible solution, satisfying the given conditions. Thus, since such a solution is built up of planes, it is always true that

[19] It is hoped that the mathematical questions which are suggested by this work can be treated more fully in a later paper. In particular, this graphical method can be so generalized as to give a solution of any kind, whether periodic or not, when its values are known along any three incommeasurably spaced, non-characteristic, parallel lines.

$$\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial^2 u}{\partial t^2},$$

and the boundary conditions and Krigar-Menzel's law are obviously satisfied. To show that Young's law is also satisfied, it is necessary to get the Fourier development of the curve $u = \phi(t)$ corresponding to some incommensurable value of x. Krigar-Menzel has proved[20] that any continuous periodic curve, whose period is made up of μ straight lines, can be developed in the form

$$u = \frac{A_0}{2} + \sum_1^\infty A_n \cos \frac{n\pi t}{T} + \sum_1^\infty B_n \sin \frac{n\pi t}{T},$$

with

$$\left.\begin{matrix} A_n \\ B_n \end{matrix}\right\} = \frac{T}{n^2\pi^2}\left[(\theta_{\mu-1}^{\mu} - \theta_0^{1})\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{n\pi\tau_0}{T}\right.$$

$$+ (\theta_0^{1} - \theta_1^{2})\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{n\pi\tau_1}{T}$$

$$\cdots\cdots\cdots$$

$$\left. + (\theta_{\mu-2}^{\mu-1} - \theta_{\mu-1}^{\mu})\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{n\pi\tau_{\mu-1}}{T}\right],$$

where $2T$ is the period, $\tau_0, \tau_1, \ldots \tau_{\mu-1}$ and $\tau_\mu (= \tau_0 + 2T)$ are the abscissae of the various corners, and $\theta_0^{1}, \theta_1^{2}, \ldots \theta_{\mu-1}^{\mu}$ are the slopes of the μ lines. Let this formula be applied to a section of any integral surface along a line $x = \epsilon$, where ϵ is a small incommensurable quantity. The left hand edge of an integral surface is always made up of the diagonals of a number of its constituent facets, and each of these planes would be cut by the plane $x = \epsilon$ in a horizontal line. Therefore, the desired section, for a case p/q, is always q long horizontal lines, at various heights, connected by q short oblique lines of various slopes.[21] Beginning with one of the oblique lines, the abscissae of the corners will have one or the other of the following sets of values, according as p is odd or even.

---

[20] Wied. Ann., 49, 545 (1893).

[21] The stair-like forms already described in section 3 are special cases of this general type of figure.

$$\tau_0\ ,\quad \tau_1\ ,\quad \tau_2\ ,\quad \tau_3\ ,\ \ldots\ \tau_\mu\,(=\tau_{2q})\ ,$$

$$-\epsilon\ ,\quad +\epsilon\ ,\quad \frac{2T}{q}-\epsilon,\quad \frac{2T}{q}+\epsilon,\ \ldots\ 2T-\epsilon\ ,$$

$$\frac{T}{q}-\epsilon,\quad \frac{T}{q}+\epsilon,\quad \frac{3T}{q}-\epsilon,\quad \frac{3T}{q}+\epsilon,\ \ldots\ 2T+\frac{T}{q}-\epsilon.$$

In either case

$$\theta_1^{\,2}=\theta_3^{\,4}=\theta_5^{\,6}=\ldots=\theta_{\mu-1}^{\,\mu}=0;$$

and furthermore

$$\theta_0^{\,1}+\theta_2^{\,3}+\theta_4^{\,5}+\ldots+\theta_{\mu-2}^{\,\mu-1}=0,$$

because the net gain in the height of the curve through a period is zero. Under these conditions, Krigar-Menzel's formula becomes

$$\left.\begin{matrix}A_n\\B_n\end{matrix}\right\}=\frac{T}{n^2\pi^2}\left[-\theta_0^{\,1}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\frac{n\pi\tau_0}{T}+\theta_0^{\,1}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\frac{n\pi\tau_1}{T}\right.$$

$$-\theta_2^{\,3}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\frac{n\pi\tau_2}{T}+\theta_2^{\,3}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\frac{n\pi\tau_3}{T}$$

$$\ldots\ldots\ldots$$

$$\left.-\theta_{\mu-2}^{\,\mu-1}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\frac{n\pi\tau_{\mu-2}}{T}+\theta_{\mu-2}^{\,\mu-1}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\frac{n\pi\tau_{\mu-1}}{T}\right]$$

$$=\mp\frac{2T}{n^2\pi^2}\left[\theta_0^{\,1}\left\{\begin{matrix}\sin\\\cos\end{matrix}\right\}\left(\frac{n\pi}{T}\frac{\tau_1+\tau_0}{2}\right)\sin\left(\frac{n\pi}{T}\frac{\tau_1-\tau_0}{2}\right)\right.$$

$$+\theta_2^{\,3}\left\{\begin{matrix}\sin\\\cos\end{matrix}\right\}\left(\frac{n\pi}{T}\frac{\tau_3+\tau_2}{2}\right)\sin\left(\frac{n\pi}{T}\frac{\tau_3-\tau_2}{2}\right)$$

$$\ldots\ldots\ldots$$

$$\left.+\theta_{\mu-2}^{\,\mu-1}\left\{\begin{matrix}\sin\\\cos\end{matrix}\right\}\left(\frac{n\pi}{T}\frac{\tau_{\mu-1}+\tau_{\mu-2}}{2}\right)\sin\left(\frac{n\pi}{T}\frac{\tau_{\mu-1}-\tau_{\mu-2}}{2}\right)\right].$$

When p is odd, this becomes

$$\left.\begin{matrix}A_n\\B_n\end{matrix}\right\}=\mp\frac{2T}{n^2\pi^2}\sin\frac{n\pi\epsilon}{T}\left[\theta_0^{\,1}\left\{\begin{matrix}\sin\\\cos\end{matrix}\right\}\left(\frac{0\,n\pi}{q}\right)+\theta_2^{\,3}\left\{\begin{matrix}\sin\\\cos\end{matrix}\right\}\frac{2n\pi}{q}+\ldots\right.$$

$$\left.+\theta_{\mu-2}^{\,\mu-1}\left\{\begin{matrix}\sin\\\cos\end{matrix}\right\}\frac{2(q-1)n\pi}{q}\right];$$

and when p is even, the form is similar. Whenever n is equal to a multiple of q, both A_n and B_n are zero, the first because each of the sine terms is zero, and the second because each of the cosine terms is one and the sum of the θ's is zero. Young's law is, therefore, always satisfied. And finally, as has been said, it can be proved that Young's and Krigar-Menzel's laws determine the motion of a finite string uniquely.[22] It is, therefore, evident that the method proposed must lead in every case to the desired solution.

A number of solutions, obtained by this method from Krigar-Menzel's law, are shown on the first and second of the accompanying plates. The third plate shows some typical $u\,t$ curves (a) as read off from the various surfaces, and (b) as reproduced experimentally by the methods of section 4.

The surfaces of the first two plates show certain serial relations which are interesting in the light of Young's and Krigar-Menzel's work on the excitation of individual partials. The aliquot cases $(1/k)$ in the first line are obviously related, and are approaching Helmholtz's major solution as a limit as k increases;[23] the amplitude for a given bowing speed is, however, becoming infinite. The mode of vibration which one would actually get in the limiting case, that is, if one were to bow at the end of the string, would be absolute rest, something quite different from the form toward which the surfaces are tending.

A similar serial relation is to be found among the surfaces of the second line. They are the cases which correspond to the various values of the expression $m/(2\,m+1)$ for integral values of m, just as those of the first line are, essentially, cases which can be grouped under the formula $m/(m+1)$.[24] The series in the second line begins with $\frac{1}{3}$ and approaches $\frac{1}{2}$ as a limit as m increases. The surfaces show a tendency toward a mode of vibration in which the middle of the string would be nearly at rest and each half would be moving in accordance with Helmholtz's major solution; but the amplitude of the motion is, as before, increasing indefinitely. In other words, all the partials which have

[22] At least within a multiplicative constant which can be determined by means of the velocity law.

[23] The integral surface corresponding to Helmholtz's major solution alone is built up of parts of hyperbolic paraboloids. Its contour lines are parts of rectangular hyperbolae with two adjacent sides of a fundamental square as asymptotes. Its appearance would be much like that of the surface for the case $\frac{1}{4}$.

[24] Any surface $(1-p/q)$ differs from the related surface (p/q) only in the signs of its displacements. To get the surface $\frac{2}{3}$ from the surface $\frac{1}{3}$ one has only to interpret each hill as a hollow and *vice versa*.

nodes at the middle of the string are coming out more and more strongly, nearly in the same proportion, as the bowing-point approaches the middle of the string. This is also brought out by the envelopes of these surfaces which are shown in Figure 9. As before, the surface at the limit ($\frac{1}{2}$) is quite different from the limit of the surfaces. It is possible to get a series converging toward $\frac{1}{2}$ from the other side by means of the formula $m/(2\,m-1)$ as is indicated in Table IV; the approach is through

TABLE IV.

	1	2	3	4	5	Limit	5	4	3	2	1	
$\frac{m}{m+1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	1						
$\frac{m}{2m+1}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$		$\frac{1}{2}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	1	$\frac{m}{2m-1}$
$\frac{m}{3m+1}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{3}{10}$			$\frac{1}{3}$		$\frac{4}{11}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{m}{3m-1}$
$\frac{2m}{5m+1}$	$\frac{1}{3}$		$\frac{3}{8}$		$\frac{5}{13}$	$\frac{2}{5}$	$\frac{5}{12}$		$\frac{3}{7}$		$\frac{1}{2}$	$\frac{2m}{5m-1}$

Showing the serial relations among the surfaces of the first and second plates.

essentially the same set of surfaces, and the same set of partials come out strongly, but in the opposite phase.[25]

Parts of similar series converging toward $\frac{1}{3}$ are shown on the second plate, the formula being $m/(3\,m \pm 1)$. They show very prettily Krigar-Menzel's change of phase in the 3d, 6th, 9th, etc. partials. The envelopes of the first of these series are shown in the upper half of Figure 11, and those of the second series on a reversed scale in the lower half of that figure. It should be compared with Figures 9 and 1, and especially with Figure 10, which is Young's representation of essentially the same fact. Finally, in the lower line of the second plate are two surfaces belonging to the convergences upon $\frac{2}{5}$, the formula being $2\,m/(5\,m \pm 1)$. In them the 5th, 10th, etc., partials are beginning to come out strongly, and are, as usual, in opposite phases.

Notwithstanding this essential difference, the surfaces for $\frac{A}{1/2}$ and $\frac{A}{1/3}$ are, in general appearance, very much alike, and a comparison of either of them with the surface for $\frac{2}{5}$ itself seems to show that each of them is

[25] See the preceding footnote.

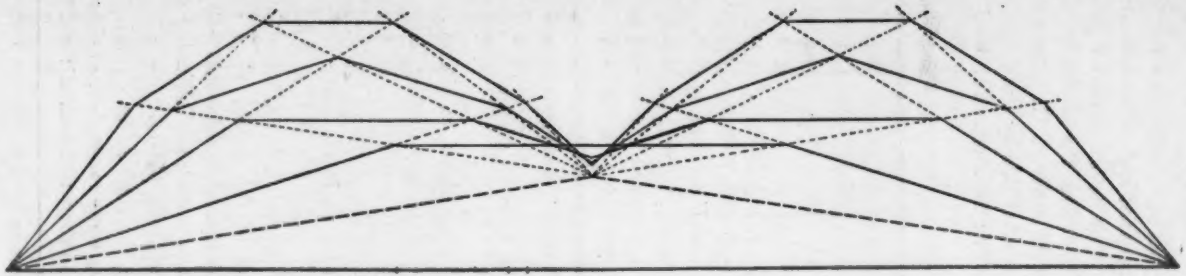


FIGURE 9. The envelopes, for a single bowing speed, of the cases $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{4}{9}$, showing that as the bowing point approaches $\frac{1}{4}$ the string seems to divide itself more and more definitely into two vibrating segments and tends to vibrate more and more violently. The amplitude of the middle point of the string is decreasing through the values $\frac{4}{3}$, $\frac{6}{5}$, $\frac{8}{7}$, $\frac{10}{9}$, etc., toward 1, which is its actual value in the limiting case. The envelope for that case is drawn with long dashes for comparison with the limit of the envelopes. The horizontal and vertical scales of this figure and of Figure 11 are proportional to those of Figure 1, with which these figures should be compared. They both can be extended indefinitely by simple generalizations of the ray principle already discussed for Figure 1, and each of them furnishes a graphical method for obtaining any envelope belonging to its convergence directly, without the labor of working out the corresponding solution as a whole.

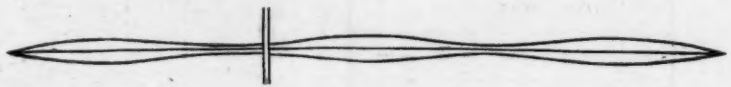


FIGURE 10. Young's figure of a violin string bowed near a principal node. This should be compared with Figure 11.

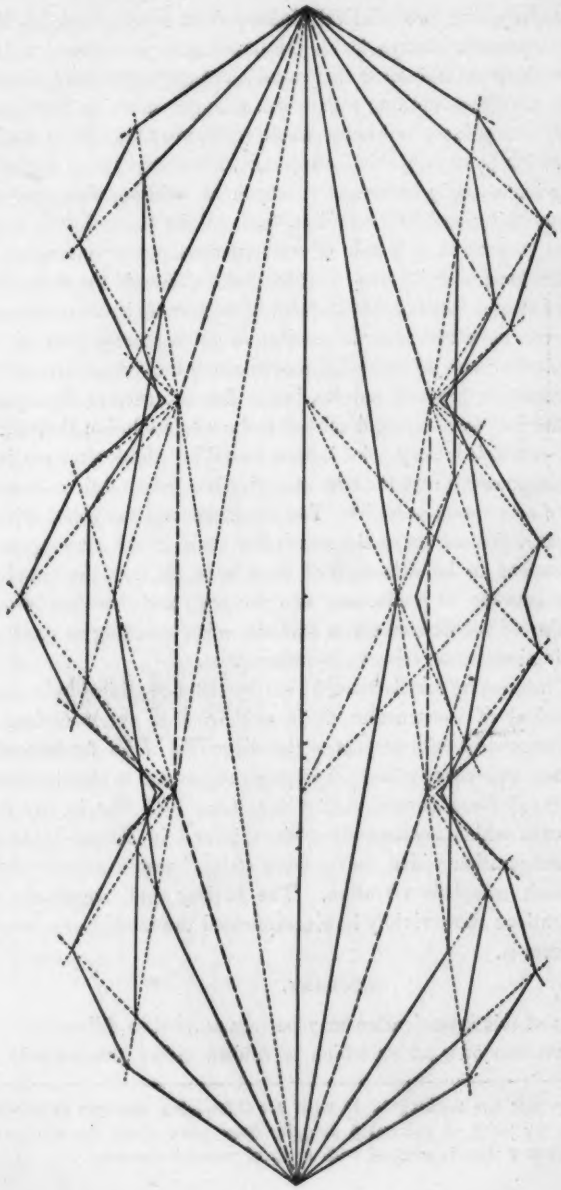


FIGURE 11. In the upper half, the envelopes for the cases $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{10}$, representing one of the convergences upon $\frac{1}{1}$. In the lower half, those for the cases $\frac{1}{1}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, representing the other convergence. The indentations are at the points $\frac{1}{4}$ and $\frac{3}{4}$ in all cases. The envelope for the limiting case $\frac{1}{1}$ itself is drawn with long dashes. (Cf. Figures 1 and 9).

the result of superposing two relatively independent components, (*a*) the 5th, 10th, etc., partials, always in the Helmholtzian proportion, which dominate the whole solution *near* the point $\frac{2}{3}$, but are completely absent *at* that point, and (*b*) something else which changes more or less continuously near the point $\frac{2}{3}$ and is the whole solution at that point itself. In the surfaces for $\frac{5}{12}$ and $\frac{5}{13}$ this *b* component has not yet been entirely masked by the increasingly important *a* component — hence their similarity. The same thing can be noticed in each of the preceding convergences, and, in general, in any pair of convergences corresponding to a formula of the form $p\,m/(q\,m \pm 1)$; so that, although the complete solution, regarded as a function of the point of excitation, is discontinuous in every interval, nevertheless it is possible to get a general idea of its changes; for as the point of excitation moves along the string, one set of partials after another becomes infinite, jumps through zero to the opposite infinity, and decreases again, dominating the whole solution through a brief interval — which is very brief indeed for all but the lowest partials — and then disappearing into the ever changing but relatively continuous *b* component of subsequent cases.[36] The significance of the points which have been arbitrarily chosen as the successive steps in the convergences of Table IV seems to be that each of them is in its turn the simplest rational point between its predecessor and the goal, and therefore shows the gradual rise of the desired set of partials, while avoiding as much as possible the dominant intervals of extraneous sets.

One other interesting fact is brought out by these graphical solutions. From the method of construction, it is evident that every surface is necessarily symmetrical with respect to the diagonals of its fundamental squares, so that any curve $y = \phi\,(t)$ for a given point x is identical with a curve $y = \psi\,(x)$ for a corresponding time $t = xT/l$; that is, any one of the $y\,t$ curves which are reproduced in this and in Krigar-Menzel's paper is a configuration which an "infinite string" would actually take on once in each complete vibration. The variety and irregularity of these configurations show vividly how complicated the motion of a bowed string sometimes is.

Summary.

The results of this investigation may be summarized as follows:

1. The envelope of a string which is rubbed either transversely or

[36] Of course this has nothing to do with the interesting question as to what happens when the point of excitation actually does move along the string, as, for instance, when a wire is stroked with a bit of rosined chamois.

longitudinally at an aliquot point $(1/k)$ is not Helmholtz's parabola itself, but k chords inscribed in that parabola; and for a given bowing speed, the relative magnitude of the envelopes of the aliquot cases is that indicated in Figure 1. These facts are proved both theoretically and experimentally. The envelopes of the simpler non-aliquot cases have also been worked out theoretically, but it has not seemed worth while to verify them experimentally.

2. Helmholtz's velocity law (see page 695) has been accurately verified for the simpler aliquot cases.

3. Krigar-Menzel's law (see page 696) has been verified for the simpler rational cases, both aliquot and non-aliquot.

4. A graphical method has been developed by means of which the differential equation of the vibrating string can be solved under any set of boundary conditions which the theory of the rubbed string can present. The advantage of this method is that it gives directly the motion of any point of the string, or the configuration of the string as a whole at any time, — information which it is difficult, even in the simplest cases, to obtain by means of trigonometric series.

5. Each of the solutions for the non-aliquot cases $\frac{2}{3}$, $\frac{2}{7}$, and $\frac{3}{7}$ obtained by this method, has been verified at a number of typical points along the string.

6. It has been shown that all rational cases can be grouped into a great number of series or convergences, of which the series of aliquot cases is the simplest example. The solutions belonging to any one of these series are so related to one another that it is possible to predict with considerable accuracy the general characteristics of an unknown solution, and, in particular, to get its envelope, merely by determining to what convergence it belongs. Also these convergences are interesting in connection with the observations of Young and of Krigar-Menzel (see page 694) on the prominence of those partials which have nodes near the point of excitation.

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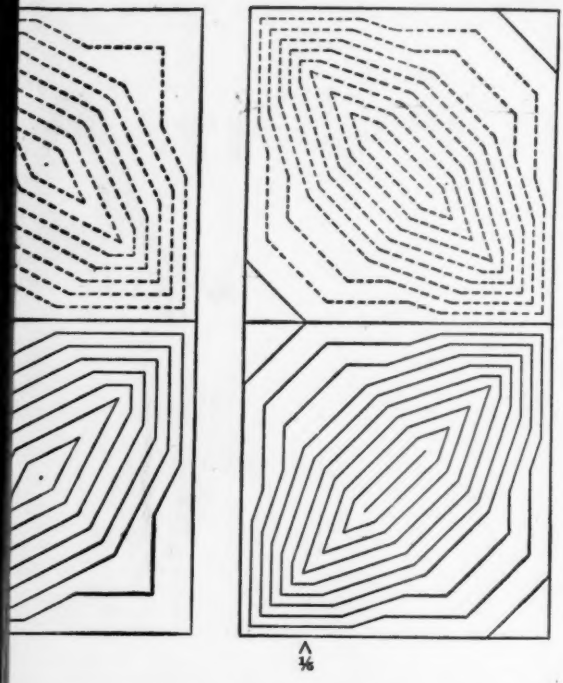
THE ALIQUOT CASES.

THE RATIONAL CASES CONVERGING ON 1/2.



The short side of each integral surface is
x axis, the long side as the t axis. All contou
are drawn with dashes. In some cases, par
$\pm \frac{1}{2}$ are dotted in to show more clearly the
face. It would not have been easy to draw
scales which corresponded to a single bowing
other comparable fashion. The number of
surface is, therefore, arbitrary, and their de
surfaces indicates only roughly the relative sl
those surfaces under similar conditions. For
of this kind the reader is referred to Figures

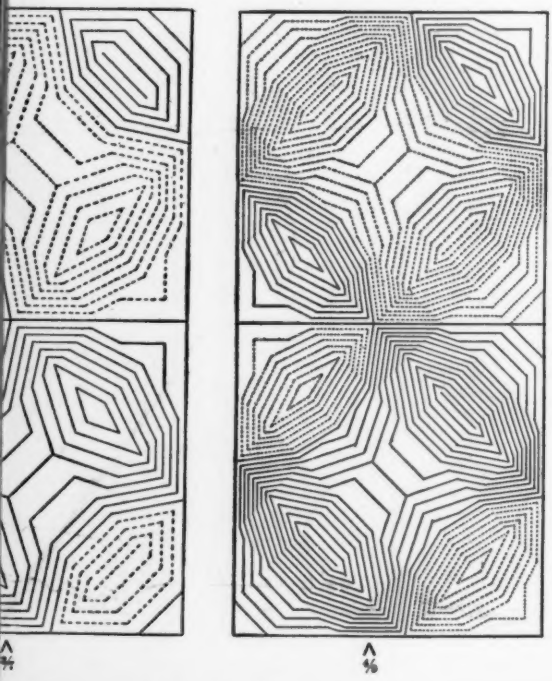
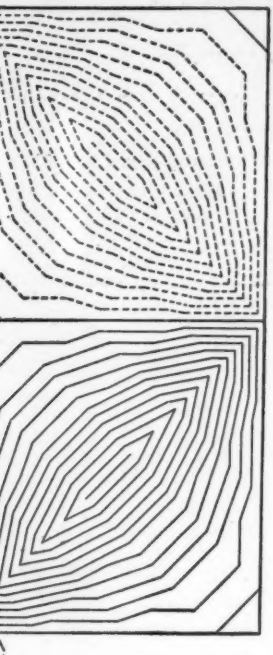
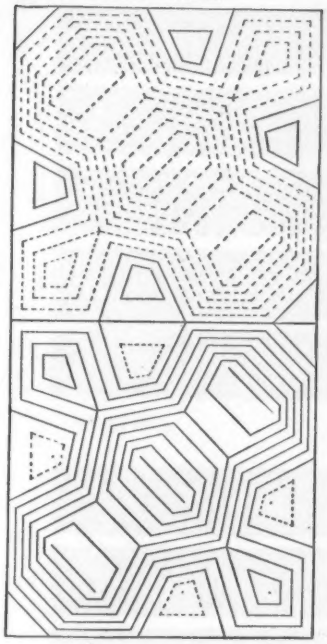


PLATE I.

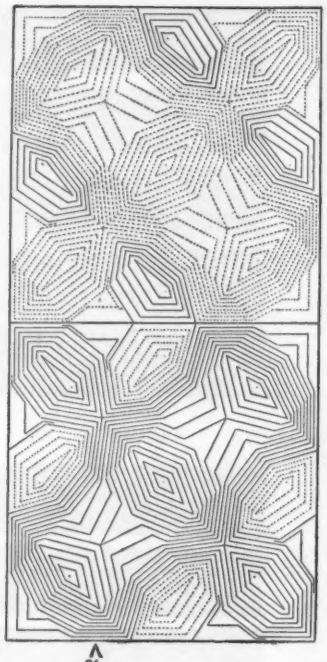


ace is to be taken as the
contours below sea-level
s, parts of the contours
y the nature of the sur-
o draw the surfaces on
owing speed, or in any
er of contours in each
eir density in different
tive slopes or heights of
For exact information
ures 1, 9, and 11.

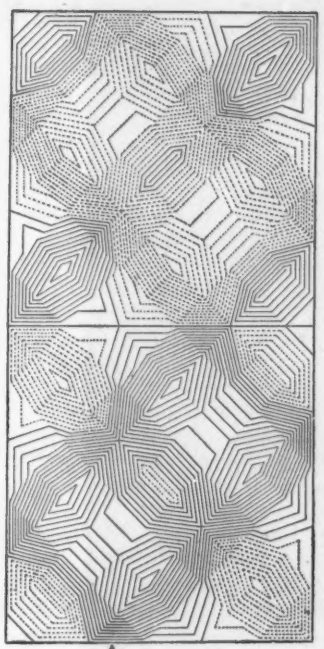
CASES CONVERGING ON $\frac{1}{2}$.



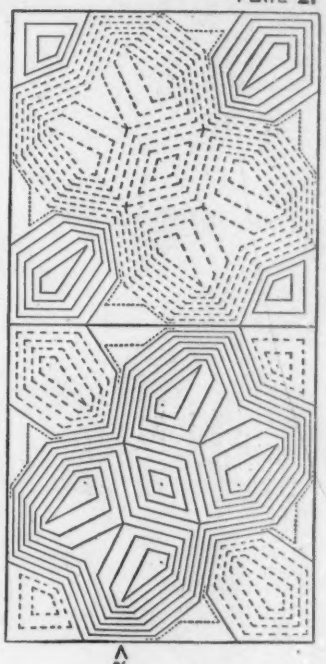
$\frac{5}{7}$



$\frac{5}{10}$

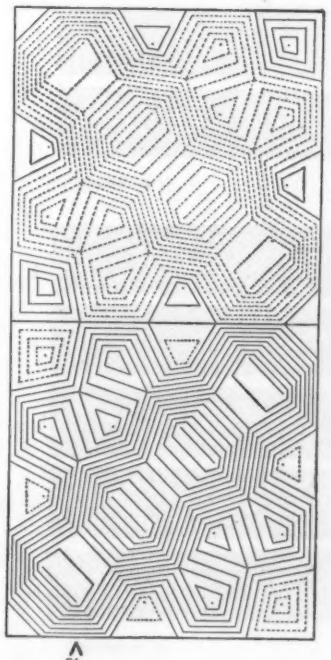


$\frac{5}{11}$



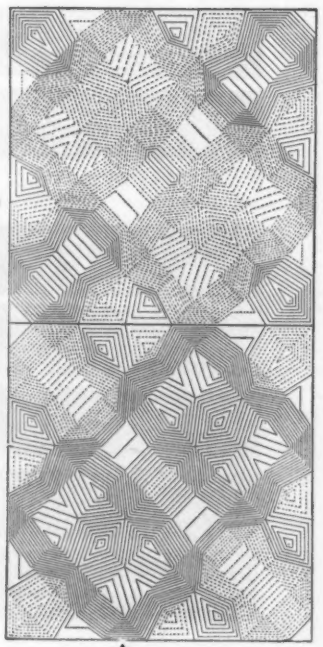
$\frac{5}{8}$

A CASE IN THE CONVERGENCE ON $\frac{1}{2}$.

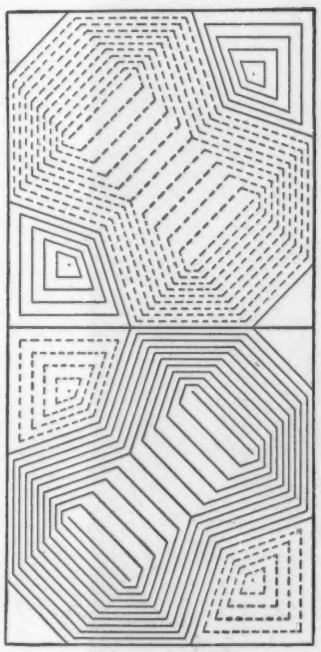


$\frac{5}{8}$

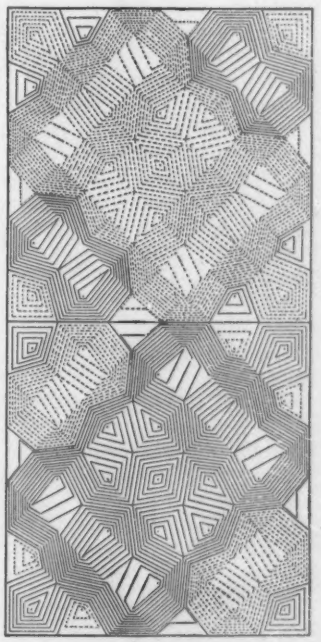
CASES CONVERGING ON $\frac{3}{5}$.



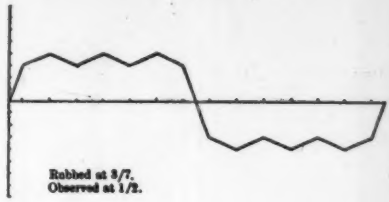
$\frac{5}{13}$

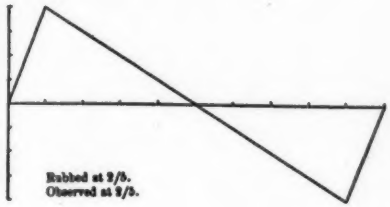


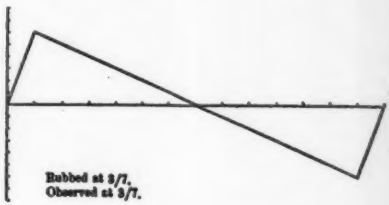
The limiting case ($\frac{3}{5}$) itself.

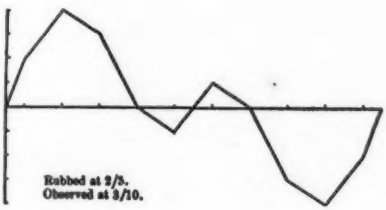


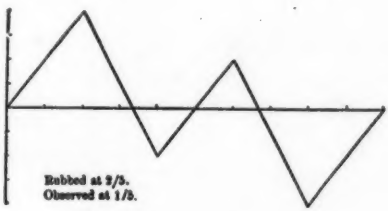
$\frac{5}{12}$

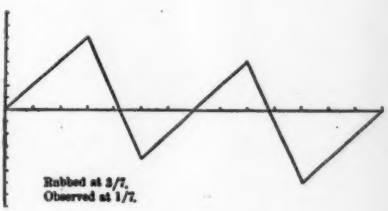


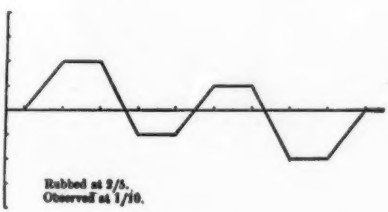


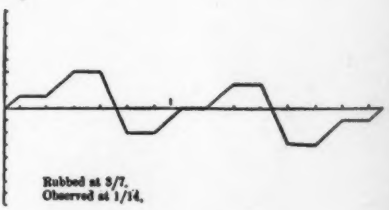




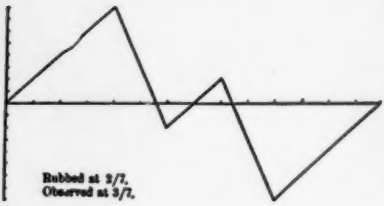


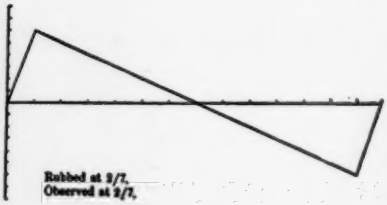


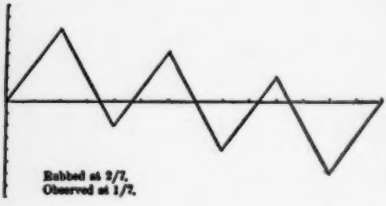


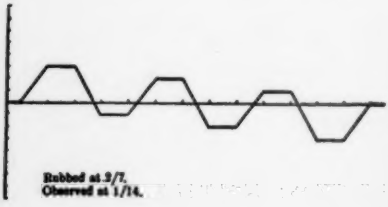


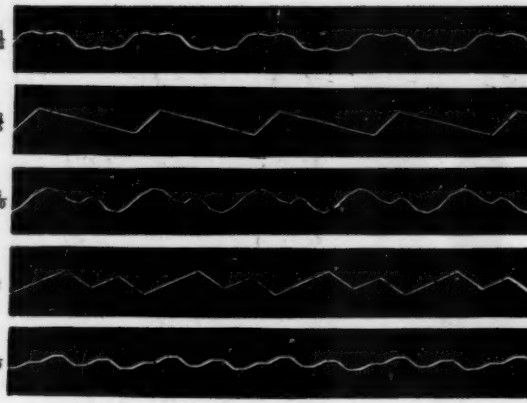
This plate shows, as functions of the time, the displacements of a numb each of the three simplest non-aliquot cases. The curves in the three colum theoretically by means of the corresponding integral surfaces. Those in th are experimental verifications, obtained by the method of section 4, and graphically. The defects are on the original slides.











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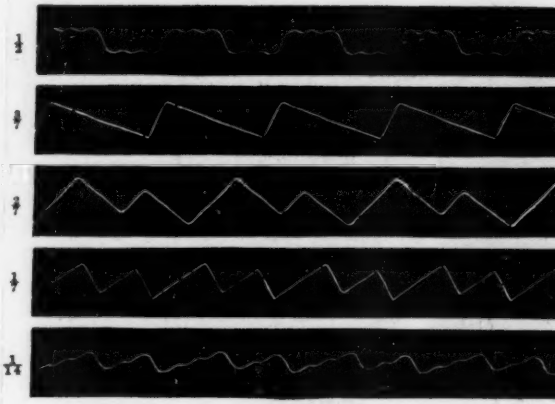

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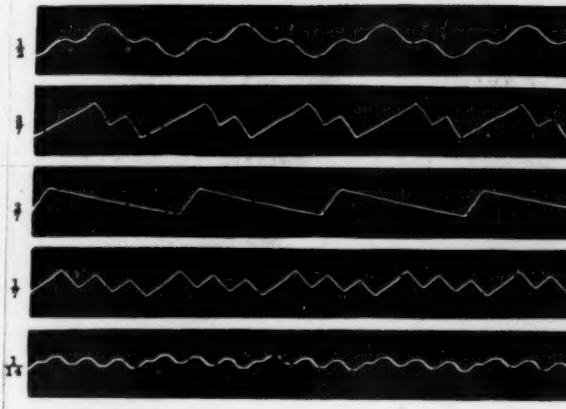

Rubbed at 2/7.

number of typical points for
columns above were obtained
in the column at the right
and much enlarged photo-
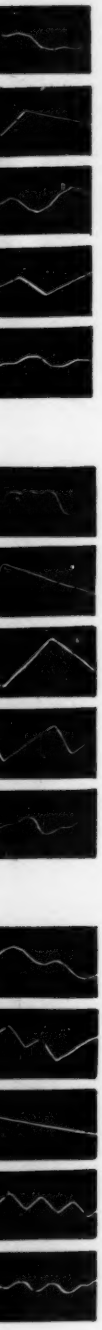
PLATE 3.